On the gravitational collapse in anti-de Sitter space-time

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Abstract

We study the semiclassical evolution of a self-gravitating thick shell in Anti-de Sitter space-time. We treat the matter on the shell as made of quantized bosons and evaluate the back-reaction of the loss of gravitational energy which is radiated away as a non-adiabatic effect. A peculiar feature of anti-de Sitter is that such an emission also occurs for large shell radius, contrary to the asymptotically flat case.

Key words: Radiating this shell, Anti-de Sitter space-time, Semiclassical Gravity
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One of the main issues arising in quantum gravity is the search for a unitary description of the process of black hole formation and evaporation [1]. The conjectured AdS-CFT correspondence may lead to a way of solving this puzzle, as it provides a gauge theory (hence unitary) description for processes occurring in string theory and asymptotically anti-de Sitter (AdS) space-times [2]. So far, however, it has been difficult to find a concrete description of such a process within this framework. For this reason, one is led to examine simplified models, for black hole formation, such as the collapse of a spherical shell of matter, in order to get an insight of what a full quantum gravity description would be. This is the path followed in Refs. [3,4,5]. The main unsolved issue in these analysis is the description of how the initial stages of the evolution, for a shell with a radius much greater then its horizon, may be related to the late stages of the collapse, when a black hole is about to be formed. In Ref. [3], it is suggested that a solution may be provided by taking into account the radiation emitted by the shell as it collapses. This is just what we are going to describe in the following.

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In a series of papers [6,7], we have studied the semiclassical dynamics of a self-gravitating shell made of bosonic matter (modelled as a set of $N \gg 1$ "microshells") which collapses in an asymptotically flat (Schwarzschild) space-time. The (average) radius of the shell is taken to be a function of (proper) time $r = R(\tau)$, and its thickness δ is small but finite ($0 < \delta \ll R$) throughout the evolution. The main result of Ref. [7] was that tidal forces acting inside the shell induce non-adiabatic (i.e., proportional to the contraction velocity) changes in the quantum mechanical states of the microshells and give rise to a probability of excitation which becomes appreciable as the shell approaches its own gravitational radius $R_H = 2\,M_s$, where M_s is the Arnowitt-Deser-Misner (ADM) mass of the shell [3]. If the bosonic microshells are coupled to a radiation field, they will then emit the excess energy and the ADM mass M_s will steadily decrease. This induces an effective (quantum) tension which will modify the trajectory of the shell radius. We explicitly determined the evolution as long as R is not too close to R_H, where some of the approximations employed break down, and showed that the shell contracts at constant (terminal) velocity.

In the present paper we want to generalize our previous results considering a shell embedded in asymptotically AdS space-time with cosmological constant $\Lambda = 3/\ell^2$ (for the thermodynamics of such a case see Ref. [8]). As we shall show below, one expects strong non-adiabatic effects not just at small shell radius (when $R \sim 2\,M_s$ and the self-gravity of the shell dominates), but also for $R \gg \ell$ (when it is the cosmological constant that mainly determines the shell motion). In particular, we will focus on R large, in order to highlight the differences with respect to the case already analyzed in Refs. [6,7].

The Schwarzschild-AdS metrics inside and outside the shell at $r = R(\tau)$ can be written as

$$ds^2 = -f_{i/o}\,dt^2 + f_{i/o}^{-1}\,dr^2 + r^2\,d\Omega^2 \ , \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$, τ is the shell proper time,

$$d\tau \equiv \sqrt{f_o(R)}\,dt = \sqrt{f_i(R)}\,dt \ , \tag{2}$$

and

$$f_{i/o}(r) = 1 - \frac{2\,M_{i/o}}{r} + \frac{r^2}{\ell^2} \ . \tag{3}$$

with $M_{i/o}$ the ADM masses for $r < R$ and $r > R$ respectively. A straightforward application of the junction conditions [9] yields the equation of motion for the shell,

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[3] With use units with $c = 1$ and $\ell_p^2 = \hbar\,G_N$ is the Planck length squared.

$$\dot{R}^2 = \left(\frac{M_\mathrm{o} - M_\mathrm{i}}{M}\right)^2 + \frac{M_\mathrm{o} + M_\mathrm{i}}{R} + \frac{M^2}{4\,R^2} - \frac{R^2}{\ell^2} - 1 \ , \tag{4}$$

where a dot denotes derivative with respect to τ. The proper mass M, as well as $M_\mathrm{i/o}$, may depend on τ [10]. In particular, we shall consider M constant [4] and initial conditions at $\tau = 0$ such that $\dot{R}(0) = 0$ [$R(0)$ equals the classical turning point]. In order to compare with the asymptotically flat case (obtained as $\ell \to \infty$), we shall also set $M_\mathrm{i} = 0$ and consider the range of parameters

$$\delta \ll M \lesssim M_s(0) \ll \ell \ll R(0) \ , \tag{5}$$

where $M_s \equiv M_\mathrm{o}(\tau)$. Note also that $R(0) \gg \ell$ requires $M_s(0)/M \sim R(0)/\ell$.

Let us now note that the proper acceleration of the shell radius given by

$$\ddot{R} = -\frac{M_s}{2\,R^2} - \frac{M^2}{4\,R^3} - \frac{R}{\ell^2} \ , \tag{6}$$

is dominated by the last term for large R. One then obtains that, for $R \gg \ell$ and near the turning point, the acceleration of the shell is much larger than it would be in asymptotically flat space ($\ell \to \infty$), so that the contraction velocity increases faster and non-adiabatic effects might occur before R approaches R_H.

The motion of a microshell in the system of $N \gg 1$ microshells can be determined from the junction equation (4). In particular, one can obtain a mean field Hamiltonian equation [6,7] for the relative displacement $\bar{r}$ of each microshell of proper mass m with respect to the average radius R of the shell, which takes the form

$$H \equiv \frac{1}{2}\,(M + m)\,\dot{R}^2 + \frac{1}{2}\,\mu\,\dot{\bar{r}}^2 + V_0 + V_m(\bar{r}) = 0 \ , \tag{7}$$

where $\mu \equiv m\,(M - m)/M$ is the effective mass of the microshell and V_0 contains all the terms which do not depend on $\bar{r}$. Since $m \ll M = N\,m$, it is consistent just to retain terms linear in m and approximate $\mu \simeq m$. Further, we are interested in the case when the shell radius is large and its thickness small [see Eq. (5)]. The time evolution of R is thus determined by

$$\frac{1}{2}\,M\,\dot{R}^2 \simeq -V_0 \ , \tag{8}$$

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[4] As in Ref. [7], we are interested in the loss of gravitational energy from the shell. Allowing for a decrease of the proper mass M would further strengthen the effect, and will be the subject of future works.

and that of $\bar{r}$ by

$$\frac{1}{2}\, m\, \dot{\bar{r}}^2 \simeq -V_m(\bar{r})\ .\tag{9}$$

The potential, to first order in $\bar{r}$, is given by

$$V_m = \frac{M_s\, m}{2\, R}\, \frac{\bar{r}}{R} \times \begin{cases} \left(1 - \dfrac{M^2}{2\, R\, M_s}\right) & \bar{r} > +\frac{\delta}{2} \\[4mm] \left(-1 - \dfrac{M^2}{2\, R\, M_s}\right) & \bar{r} > +\frac{\delta}{2}\ . \end{cases}\tag{10}$$

Up to corrections of order $\bar{r}^2/\ell^2$, which are negligible in the range of parameters (5), the above expression is identical to the asymptotically flat case treated in Ref. [7] and represents the tidal force responsible for the microshell confinement. This is expected in a proper time formalism, since the binding potential V_m just depends on the local geometry.

The Hamiltonian constraint (9) can be quantized and, via the Born-Oppenheimer approximation, on assuming negligible quantum fluctuations of the metric, it leads to the Hamilton-Jacobi equation (8) for the average radius R and to a Schrödinger equation for the microshell [5],

$$i\,\hbar\,\partial_\tau|\, n\,\rangle = \left[-\frac{\hbar^2}{2\, m}\, \partial_{\bar{r}}^2 + V_m\right]|\, n\,\rangle\ ,\tag{11}$$

in which the potential V_m is time-dependent (recall that R and M_s are in general functions of τ). Such equation was analyzed in Ref. [7] and the low lying modes in the spectrum,

$$\langle\, \tau, \bar{r}\mid n\,\rangle = e^{-i\,n\,\Omega\,\tau}\, \Phi_n(\bar{r})\ ,\tag{12}$$

were obtained by means of quantum invariants [12]. The low energy levels are given by

$$E_n - E_0 \simeq n\,\hbar\,\Omega\ ,\tag{13}$$

where the fundamental frequency is

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[5] Such a treatment is rather lengthy and we omit the details for the sake of brevity. For the general formalism see Ref. [11]; its application to the shell model can be found in Ref. [6,7].

$$\Omega = \frac{1}{R} \sqrt{\frac{M_s}{\delta}} \, , \tag{14}$$

and the spatial width of the levels is

$$\delta \sim \ell_m^{2/3} \, R^{1/3} \left(\frac{R}{M_s}\right)^{1/3} \, , \tag{15}$$

$\ell_m = \hbar/m = N \, \ell_p^2/M$ being the Compton wavelength of the microshells.

Due to the time-dependence of V_m, the amplitude of excitation from the ground state Φ_0 to states of even quantum number Φ_{2n} at the time $\tau > 0$ is not zero. To leading order in $\dot{R}$, it is given by

$$A_{0 \to 2n}(\tau) \simeq (-i)^n \, \frac{\sqrt{(2\,n)!}}{6^n \, n!} \left(\frac{\delta}{M_s}\right)^{n/2} \dot{R}^n \, . \tag{16}$$

This expression rapidly vanishes for large R ($\gg M_s$) in the asymptotically flat case. However, in AdS, $|\dot{R}|$ grows faster for $R \gg \ell$ [as can be inferred from Eq. (6)] and leads to a non negligible amplitude in such a regime.

The microshells can be coupled to an external scalar field φ by an interaction Lagrangian density of the form

$$\hat{L}_{\text{int}} = e \sum_{n,n' \neq n} e^{-i\,(n-n')\,\Omega\,\tau} \, \Phi_n(\bar{r}) \, \Phi_{n'}(\bar{r}) \, \hat{\varphi}(t, R + \bar{r}) \, , \tag{17}$$

where e is the coupling constant and $\tau = \tau(t)$ [see Eq. (2)]. The field φ satisfies the Klein-Gordon equation $\Box \varphi = 0$ everywhere in AdS. However, since we are interested in the flux of radiation outgoing from the shell, we just need solutions outside the shell and neglect the time-dependence of R (consistently with the approximation in which we keep the leading term for small shell velocity). The scalar field for $r > R$ is given by a sum over the normal modes [3]

$$\langle t, r, \theta, \phi \mid n, l, m \rangle = \frac{e^{-i\,\omega_{nl}\,t}}{r} \, u_{nlm}(r) \, Y_l^m(\theta, \phi) \, , \tag{18}$$

where the Y's are standard spherical harmonics. The radial functions are more easily expressed in terms of the turtle-like coordinate $dr_* \equiv dr/f_{\text{o}}$, and read

$$u_{nlm} = c_{nl} \, \cos^2(r_*) \, \sin^{1+l}(r_*) \, P_n^{l+\frac{1}{2},\frac{3}{2}} (\cos(2\,r_*)) \, , \tag{19}$$

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where the c_{nl}'s are normalization constants, the P's Jacobi polynomials and

$$\omega_{nl} = \sqrt{\frac{f_{\rm o}(R)}{1 + R^2/\ell^2}} \, \frac{2\,n + l + 3}{\ell} \; . \tag{20}$$

We may now compute the transition amplitude for the entire process during which a microshell is excited from the ground state Φ_0 to an excited state Φ_n and then decays back to Φ_0 by emitting scalar quanta $|\, n, l, m \,\rangle$,

$$\mathcal{M}_n(t, 0) \equiv \frac{i}{\hbar} \int_0^t dt' \sum_{i=1}^{N} \int d\bar{r}'_i \, \Phi_0^*(\bar{r}'_i) \, \Phi_0(\bar{r}'_i) \, \langle\, n, 0, 0 \, |\hat{L}_{\rm int}(t', \bar{r}'_i)|\, 0, 0, 0 \,\rangle \; , \tag{21}$$

where we enforced the spherical symmetry (so that only states with $l = m = 0$ are included) and summed over the N microshells located at $R + \bar{r}_i$. We further assume that $A_{0 \to 2}(t)$ is the only non-negligible excitation amplitude (that is, we neglect the possibility that each microshell can be excited more than once) and consider a symmetrized state for the N microshells. The probability of emitting an energy equal to $2\,\Omega$ during the interval between $t = 0$ (when $\dot{R} = 0$) and t,

$$P(2\,\Omega; t) \simeq \sum_{n' \, n''} \mathcal{M}^*_{n'}(t, 0) \, \mathcal{M}_{n''}(t, 0) \; , \tag{22}$$

contains the projection of the Wightman function onto states with $l = m = 0$,

$$\langle\, \hat{\varphi}(R') \, \hat{\varphi}(R'') \,\rangle = \sum_n \frac{e^{-i\,\omega_{n0}\,(t' - t'')}}{2\,\omega_{n0}\,R'\,R''} \, u_{n00}(R'_*) \, u^*_{n00}(R''_*)$$

$$= \sum_n \frac{e^{-i\,\omega_{n0}\,(t' - t'')}}{2\,\omega_{n0}\,R'\,R''} \, s_n^2(R', R'') \; , \tag{23}$$

and is finally given by

$$P(2\,\Omega; t) \sim \frac{e^2\,N^2}{\hbar^2} \, |A_{0 \to 2}|^2 \sum_n s_n^2(R, R) \, \frac{\sin^2\left[\left(2\,\Omega\,\sqrt{f_{\rm o}} - \omega_{n0}\right) t/2\right]}{\omega_{n0}^2 \left(2\,\Omega\,\sqrt{f_{\rm o}} - \omega_{n0}\right)^2} \; , \tag{24}$$
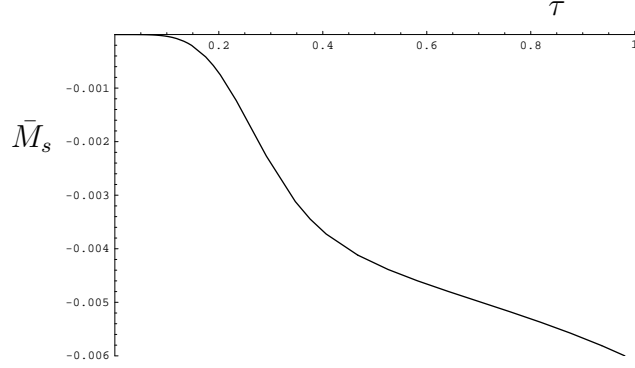
In the above expression, all functions must be evaluated at the time t and we also used the fact that the states Φ_n have a very narrow width δ [according to Eq. (5)] and can be approximated as $\Phi_n \sim \delta(\bar{r})$. Note that the lapse t should be chosen short enough so that R remains approximately constant during
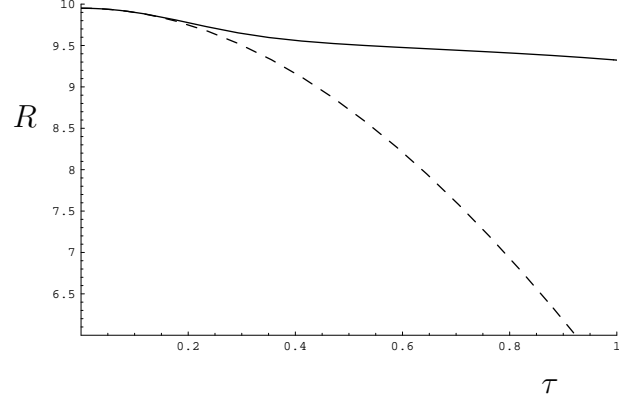
(a)
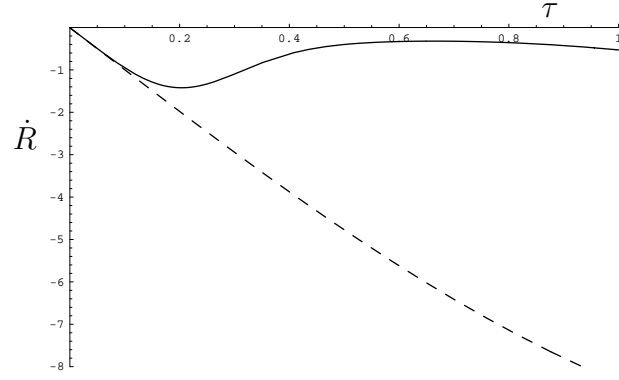
(b)

(c)

Fig. 1. (a) shell radius with (solid line) and without (dashed line) radiation; (b) shell velocity with (solid line) and without (dashed line) radiation; (b) shell ADM mass $\bar{M}_s \equiv M_s - M_s(0)$.

the corresponding time interval, but long enough to average over transient effects [6].

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[6] This is the approximation which yields the Planckian spectrum at the Hawking temperature for $R \sim 2\, M_s$ [7].

δ	M	M_s	ℓ	R	N	n	Ω	$\dfrac{|A_{0\to2}|^2}{\dot{R}^2}$	$\dfrac{\Delta M_s}{M_s}$
10^{23}	10^{24}	10^{25}	10^{26}	10^{27}	10^{44}	10^2	10^{-24}	10^{-2}	10^{-7}
10^{25}	10^{26}	10^{28}	10^{30}	10^{32}	10^{46}	$4\cdot10^2$	10^{-28}	10^{-3}	10^{-12}
10^{26}	10^{29}	10^{31}	10^{33}	10^{35}	10^{49}	$4\cdot10^4$	10^{-29}	10^{-5}	10^{-15}
10^{27}	10^{32}	10^{34}	10^{36}	10^{38}	10^{52}	$4\cdot10^6$	10^{-30}	10^{-7}	10^{-18}

Table 1

The parameters ℓ, M and N and relevant quantities at $\tau = 0$ for $\ell_m = 10^{20}$. The integer n identifies the AdS mode corresponding to the excited energy gap 2Ω and the change ΔM_s is estimated assuming $\dot{R}^2 \sim 1$ and neglecting the time of emission. All lengths are in units of ℓ_p.

For sufficiently large t (or τ), one can define a probability per unit proper time

$$\mathcal{P}_{2\Omega}(R, \dot{R}) \equiv \frac{P(2\,\Omega;\tau)}{\tau} \simeq e^2\,\dot{R}^2\,F(R, M_s)\,\delta\left(2\,\Omega - \omega_{n0}/\sqrt{f_\mathrm{o}}\right)\,, \qquad (25)$$

where the function F can be straightforwardly deduced from Eq. (24). The above expression shows that emissions occur when some frequency $\omega_{n0}/\sqrt{f_\mathrm{o}}$ (as measured in the shell frame) of an AdS mode is close to the proper energy gap $2\,\Omega$, that is

$$n^{3/2} \sim \frac{M\,M_s}{N\,\ell_p\,R}\,. \qquad (26)$$

From the probability (25) the energy emitted per unit time is straightforwardly obtained as

$$\dot{M}_s = -8\,\pi\,R^2\,\Omega\,\mathcal{P}_{2\Omega}(R, \dot{R})\,, \qquad (27)$$

which, together with the dynamical equation (4) constitutes the system of coupled equations that we solved numerically. In Fig. 1 we plot an example of the radius, velocity and ADM mass of the shell for a choice of the parameters particularly suited to obtain neat plots. It is clear that the velocity approaches a terminal value, although our approximations are not entirely reliable for too long times. For more realistic values of the parameters, it suffices to estimate some relevant quantities at $\tau = 0$ that we give in Table 1. It is clear from those figures that the larger ℓ, the smaller is the effect (at small times).

To summarize, we have analyzed the semiclassical dynamics of a shell made of bosons in AdS, thus generalizing the results previously obtained in asymptotically flat space-time [7]. As in the latter case, we find that the shell spon-

taneously emits gravitational energy in the form of radiation when its velocity of contraction is not negligible and the adiabatic approximation fails. In AdS this occurs not only for the shell radius R approaching the gravitational (Schwarzschild) radius $R_H \sim 2\,M_s$ but also for relatively large radii, $R \gg \ell$, thus increasing the importance of such an effect. Another peculiarity of AdS is that the spectrum of massless radiation is not continuous, and one therefore finds that the emission occurs as a resonance between the shell inner energy gaps and the AdS levels. Our approach therefore suggests that the quantum nature of the collapsing matter implies the existence of radiation and possibly strong backreaction. Hence, the choice of vacua in Ref. [3] for a shell radius much larger than the horizon radius (Boulware vacuum) and for a shell approaching the horizon (Unruh vacuum) may actually be connected by the evolution of the system when the backreaction is properly considered. Since the radiation is present from the very initial stages of the collapse, this situation is analogous to the "topped-up" Boulware state used in Ref. [13] with a radiation of dynamical origin.

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